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Branch-height distribution in diffusion-limited deposition

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We analyze diffusion-limited aggregation (DLA) with a branchless needle model. We modify the growth rules of our needles by assigning them a fractal dimension of $D_b \approx 1.7$, the fractal dimension of DLA. We then construct a mean-field theory of the evolution of the number of needles having particular heights. Our model accounts for the correlations within a needle. We argue that DLA is an isotropic fractal with a scaling density profile and that the fractal dimension of the individual branches should be the same as the dynamical dimension of the aggregate.

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I. INTRODUCTION

The diffusion-limited aggregation [1] model (DLA) has so far resisted any analytic solution. Attempts have been made to grasp the essential physics of the problem by breaking it into parts. We can imagine DLA to be characterized by two processes: creation of branches through tip splitting and the death of branches through competition. Tip splitting can be understood locally, but competition, a nonlocal phenomenon, is less well understood.

Several groups have tried to consider the effects of competition alone through needle models, where growth is constrained to occur only at the moving tips. Derrida and Hakim have studied the original DLA geometry of needles growing radially from a point seed through conformal maps [2]. Meakin, Cates, and Rossi studied needles growing vertically from a hyperplane seed through simulation and mean-field theories [3–5]. We present here a model which by treating competition alone, has been simplified enough to allow solution and yet still makes successful predictions about DLA.

In the commonly used electrostatic analogy for diffusion limited growth [1], the average mass flux is replaced by an electric field which develops near the deposit, which is replaced by a grounded conductor. The points of a thin grounded needle would develop an infinite electric field near the point, i.e., an infinite mass flux. References [3–5] deal with this difficulty by assigning the tip of a needle a constant radius of curvature a. The growth rate would then be

$$\dot{h} \sim a^{-1}. \tag{1}$$

However, the branches in a simulation of diffusion-limited deposition [6] do not have constant radius of curvature; larger branches have a larger radius of curvature than smaller branches. In addition, since the aggregates are fractal, not compact, there is no reason to expect (1) to apply.

One of the limitations of those needle models in which a is constant is that they predict exponentially increasing growth velocities because as needles grow, they die through competition. Each living needle captures ever more flux and grows faster [4]. Kassner notes that it ought to destroy the slow growth hypothesis [7]. A theory without the slow growth hypothesis would lose the link between DLA and many natural phenomena such as dielectric breakdown and viscous fingering. By making a a function of needle height, we can slow down the large needles and keep the whole aggregate growing smoothly.

II. CONSTRUCTION OF THE MODEL

Diffusion-limited deposition (DLD) [6] differs from DLA in that the seed is a horizontal line and particles are released from above. DLD represents the limit in DLA of a small slice near the surface of a large aggregate. A DLD aggregate is a collection of "trees" of different heights. Since each of these trees is longer than it is wide, we can model it as a needle of the same height, but zero width. We assume that the effect of a tree on other trees through shielding is roughly the same as that of a needle of the same height. We will show that this simplification does not alter the results of the simulation too much. In addition, equating needles with actual DLD branches solves the two problems facing needle models mentioned above, the singularity at the tip of the needles and the predicted exponential growth of DLD. We assign each needle the growth rate per unit flux of the branch that it represents.

We can find the growth rate per unit flux by using the fractal nature of the aggregate. If a needle has a fractal dimension D_b, and its mass increases at a certain flux rate $\partial m/\partial t$, its height $h \sim m^{1/D_b}$ should increase as

$$\partial h/\partial t \sim (\partial m/\partial t)/h^{D_b-1}. \tag{2}$$

Now, instead of counting flux at the tip only, we count flux all along the length of the branch. We assume that the tip radius will adjust itself to whatever is needed to maintain Eq. (2).

Our method of needle modeling also has applications to dendritic crystals [8] which grow under DLA-like conditions, but for which surface tension anisotropy creates a different morphology. In such dendrites, a main branch with an approximately parabolic tip develops side branches. Because of anisotropy, these side branches grow at some fixed angle to the main branch and surface tension keeps these side branches compact. We model side branches as needles. When a random walker lands at the tip of the main branch, the main branch grows and creates a new side branch. If ordinary needles [4,3] are used, the exponentially growing side branches quickly overwhelm the main branch. If we assign the needles a dimension greater than one, then the side branches are better behaved as shown in Fig 1.

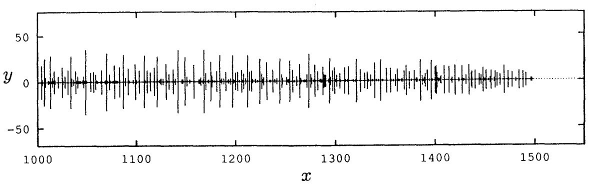


FIG. 1. The tip region of a simulated needle model dendrite. The image has been reflected across the x axis. Walkers are released from a circle and random walk to the aggregate. If they land at the tip of the main branch (along the x axis) the tip grows, creating a new side branch. If they land on a side branch, the side branch grows as a needle of dimension 2 [9]. Compare this image with Fig. 1 in Ref. [8].

III. SIMULATION

In what follows, x represents the direction parallel to the substrate, y the height above the substrate, t the time and also the mass of the aggregate since particles are introduced at a constant rate, $h(x)$ the height of the branch whose root is at $(x,0)$, $n(y,t)$ is the number of branches with $h \geq y$ at time t, and $u(x,y)$ is the probability of finding a walker at (x,y).

The model was simulated in 1+1 dimensions. We place a seed 1000 pixels wide at $y = 0$. A particle is released with uniform probability at $y = h_{\max}$, slightly above the highest point on the aggregate. It then random walks on lattice to the aggregate by calculating the distance to the aggregate and then jumping uniformly to the perimeter of a circle whose radius is that distance. If the particle jumps above $h_{\max}$, it is brought back to $h_{\max}$ with a Green's-function jump [10]. Once the particle reaches the aggregate, it is absorbed and a new particle is released. When the particle lands anywhere on a needle, it increments a counter assigned to that needle. If that counter reaches $h(x)^{D_b - 1}$ then the needle grows one unit and the counter is reset to zero.

As can be seen from Fig. 2, the model successfully reproduces the branch density profile of DLD. The branch density profiles of both DLD and our needle model are scaling functions of time which fit $n(y,t) = y^{-\alpha} f(\frac{y}{t^\gamma})$ with $\alpha = 0.7$, $\gamma = 1.1$.

Our result $\alpha = 0.7$ implies that DLD is actually self-affine, not self-similar. α describes the way branch height scales with branch number. Since branch number scales linearly with the width of an aggregate, α also describes the way height scales with width. y scales as x^α. This difference in scaling between the vertical and horizontal directions of DLD, or between the radial and azimuthal directions of DLA, has been noted in the past [11].

Some recent papers, however, have claimed that this anisotropy is the result of a slow crossover [12]. If this is true, in the limit of large aggregate size, DLA becomes an isotropic fractal with a scaling form and a fractal dimension of ≈ 1.7.

Mass is injected into the system at infinity at a constant rate. Therefore, $t \sim m$. Since a typical length scale $y \sim t^\gamma$, we have $1/\gamma = D_d - 1$ where D_d is the dynamical dimension of the aggregate as a whole. We therefore expect the large aggregate limit exponents to be

$$\alpha = 1, \quad \gamma \approx 1.4. \tag{3}$$

A theory should yield these long-time exponents, not the short-time exponents presented above.

Although this crossover is documented, its causes are not yet known. We therefore cannot directly prove that our model will mimic DLA in the long-time regime. However, the fact that the two give such similar results in the short-time regime suggests that our initial assumption is sound, namely, that a complex long object like a DLA branch will effect the diffusion field at other branches in the same way as a needle of the same height.

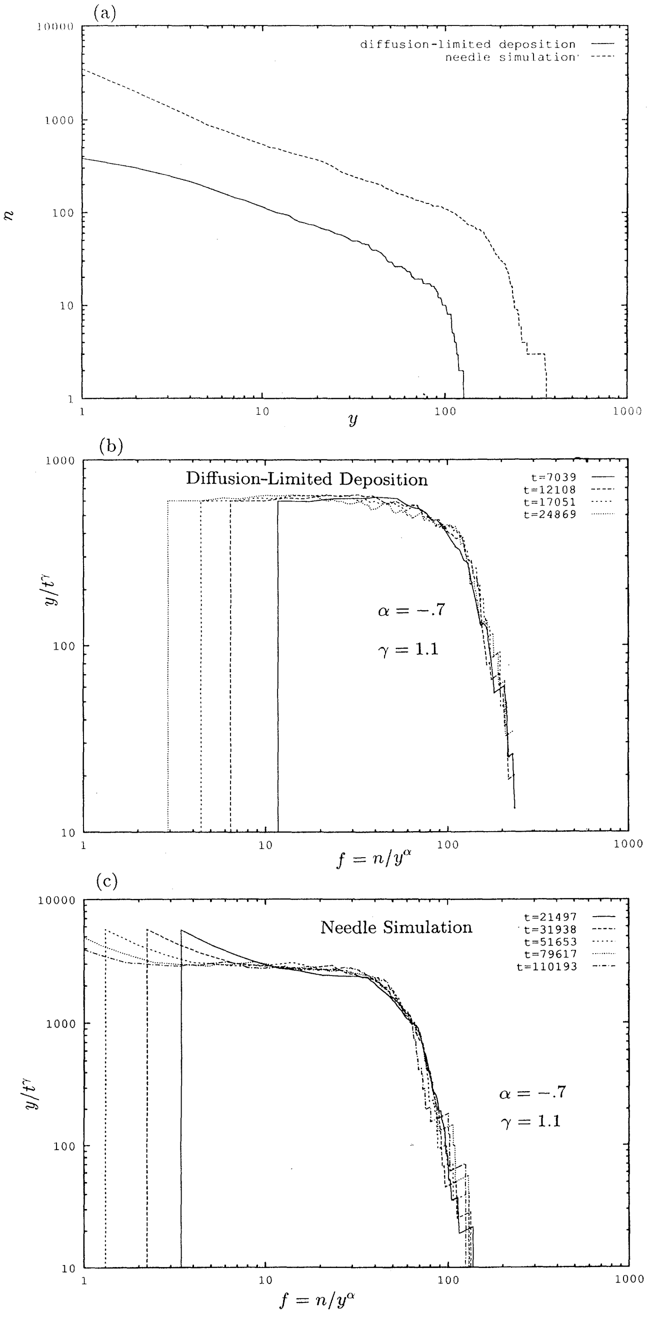


FIG. 2. Branch density profiles of (a) diffusion-limited deposition and a needle simulation. Scaling of branch density profiles in (b) diffusion-limited deposition and (c) a needle simulation.

IV. MEAN-FIELD THEORY

Following Cates [4], we can write down a one-dimensional mean-field theory to describe $n(y,t)$ and $u(y,t)$. We choose units such that various constants are unity. We start with a diffusion equation for u with some trap density n_{eff}. The relationship between n and n_{eff} will be discussed shortly:

$$\partial^2 u/\partial y^2 - u n_{\text{eff}} = \partial u/\partial t. \tag{4}$$

A mean-field expression such as this assumes that there are no significant correlations in the x direction. We assume that correlations *between* branches, e.g., that large branches tend to stunt neighboring branches, are unimportant. However, a strong correlation which we cannot ignore is that all points in the same branch have the same x position. The tip of a branch will more effectively shield the points below it when all the points on the branch are lined up instead of smeared out in the mean-field manner (as shown in Fig. 3).

The trap density n_{eff} is a term which describes the way real branches behave in a mean-field theory. Due to shielding, we posit that each branch has some effective trap density strongest toward the tip and decreasing as one moves away from the tip. Thus

$$n_{\text{eff}}(y) = \int_y^\infty dh(\partial n/\partial h)\xi(h, y/h). \tag{5}$$

Here, $\partial n/\partial h$ is the tip density at h.

We make the slow growth hypothesis that the diffusion field has time to relax after each growth step, and we replace the diffusion field with a Laplacian field,

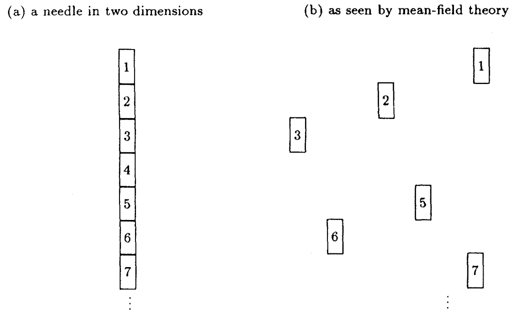
$$\partial u/\partial t = 0. \tag{6}$$

For the growth rule, we increase the mass of a branch when a walker meets it, and that branch grows at its tip at a rate determined by Eq. (2),

$$(\partial n/\partial t)(h) = h^{1-D_b} \int_0^h dy(\partial n/\partial h)u(y)\xi(h, y/h). \tag{7}$$

Boundary conditions [4] state that all walkers are absorbed before they reach the seed plane at $y = 0$ and that walkers are injected into the system at $y \to \infty$ at a constant rate,

$$u(0,t) = 0, \quad \lim_{y\to\infty} \partial u/\partial y = \text{const}. \tag{8}$$


(a) a needle in two dimensions

(b) as seen by mean-field theory

FIG. 3. (a) is a typical needle. (b) shows how this needle is seen by a mean-field theory. x positions no longer matter, so the needle behaves in a smeared-out fashion.

On the basis of the simulation, we assume scaling solutions of the form

$$\begin{aligned} n(y,t) &= y^{-\alpha} f(y/t^\gamma), \\ u(y,t) &= y^\beta g(y/t^\gamma), \quad \xi(h,\theta) = h^\nu k(\theta). \end{aligned} \tag{9}$$

Here, θ is the dimensionless variable y/h. By (8), $\beta = 1$.

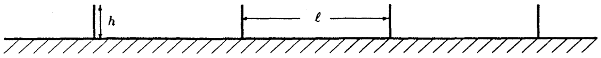


FIG. 4. The system used to calculate ξ.

We can solve for ξ by examining a system which (a) can be solved exactly, (b) satisfies the same boundary conditions as the problem at hand, and (c) involves needles. One such system is an ensemble of needles, equally spaced, attached to a plate (Fig. 4). Let ℓ be the distance between needles and h be the height of the needles and $\langle\,\rangle$ represent the average over x. Then we can rewrite Eq. (4) for the situation as

$$\xi(h, y/h) = \ell\langle\partial^2 u/\partial y^2\rangle/\langle u(y)\rangle. \tag{10}$$

In the limit $l/h \to \infty$, each spike has a negligible effect on its neighbors and can be treated as an isolated spike. A single spike on a plate can be solved using a conformal map. The complex map $z = (w^2 - h^2)^{1/2}$ takes the real axis $\Im(w) = 0$ to the real axis plus the spike $(0, ih)$. After some algebra, we find that ξ scales as h^{-1} or $\nu = -1$.

We now substitute the scaling form (9) into Eqs. (4)–(8) and derive the remaining scaling exponents,

$$\alpha = 2 + \nu = 1, \quad \beta = 1, \quad \gamma = (D_b - \nu - 2)^{-1} = 1.4. \tag{11}$$

V. DISCUSSION

For both our small DLD simulations and the needle simulation, we got $\alpha = 0.7$. In keeping with the formulas of (11), we expect the correlation exponent ν to be -1.3 if it is applicable. $\nu = -1.3$ implies $\gamma = (D - \nu - 2)^{-1} = 1.1$, also matching simulation. Thus α and γ are linked through ν.

We see from Eqs. (3) and (11) that our mean-field results agree with the large-aggregate results. We also tested the hypothesis that the discrepancy between the simulation results and the mean-field results may be due to correlations between branches as opposed to correlations between points in the same branch. We can remove these correlations by scrambling the branches after every collision. We retain the distribution of n branches of height y, but assign each branch a random x location. When this system is simulated, we recover the mean-field exponents which are also the large-aggregate exponents. This result suggests that these interbranch correlations may be mechanism which holds off the large-aggregate regime.

As we earlier decided, $\gamma = 1/(D_d - 1)$ where D_d is the dynamical dimension of the aggregate as a whole. We also see from (11) that $\gamma = 1/(D_b - 1)$ where D_b is the fractal dimension of an individual branch which implies that $D_d = D_b$.

VI. CONCLUSION

We have attempted to redefine the way in which needles are perceived as theoretical stand-ins for DLA branches. In our treatment, the growth rate of a needle is not just the flux of the tip but is also related to the nonconstant radius of curvature of the tip. This rate can be found using the fractal dimension of the aggregate. Real needles have absorbing sides. When sides are absorbing, the correlations forced by the needle geometry cannot be ignored. These can be examined by analyzing the scaling in the case of a single needle.

In constructing a needle mean-field theory, we are able to create a theory which concentrates on the number of branches instead of mass. Such a theory allows us to make comments on anisotropy as well as the fractal dimension and the width of the interface. Our theory is in agreement with measurements on large DLA aggregates, that DLA is an isotropic fractal, with a scaling profile, and the fractal dimension of the branches is the same as the dynamical dimension of the aggregate.

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